APPLICATION OF A FOKKER–PLANCK EQUATION TO DESCRIBE PROCESSES OF COAGULATION AND DISINTEGRATION OF DROPLETS IN A TURBULENT FLOW

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An analytical solution to a Fokker–Planck equation is obtained which describes evolution of the droplet distribution function in a turbulent flow with account of their coagulation and disintegration.

Turbulent flow of petroleum emulsions in tubes is characterized by complex physical phenomena of coagulation and disintegration of droplets, to which many works [1-3] are devoted, making it possible to establish specific features of these phenomena and the change in the mean dimension of the droplets. At the same time for practical computations our main interest is in the evolution of the droplet dimension distribution function with account of coagulation and disintegration described by complex stochastic integrodifferential equations, because the indicated phenomena have a random discrete character. However, if the coagulation and disintegration processes are characterized by the mean-static change in dimensions of the droplets in a certain time interval, then application of the stochastic Fokker–Planck equation [4] is more practical and attractive. As noted in [3], continuous enlargement of droplets in a turbulent flow due to turbulent diffusion is described by the equation

$$\frac{da}{dt} = \frac{4kD_{\text{T}}\varphi}{a} = \frac{m_R}{a}, \quad dt = \frac{dl}{u_0}, \tag{1}$$

where $m_R = 4kD_{T\varphi}$. At the same time, becoming larger, a droplet is specified by a large unstable surface, corresponding to the critical droplet dimension (which becomes unprofitable from an energetic viewpoint), and its disintegration [2] occurs.

Disintegration of droplets in a turbulent flow is a result of the action of the difference in dynamic heads, whose initiation sources are small-scale pulsations and surface tension. The droplet disintegration rate in the first approximation is assumed to be proportional to its dimension [1], i.e., $da/dt \sim a^n$. Then the change in mean dimensions of the droplets due to their coagulation and disintegration may be given in the form

$$\frac{da}{dt} = \frac{m_R}{a} - k_R a^n, \tag{2}$$

where k_R is the disintegration coefficient.

The solution of (2) under the prescribed initial condition

$$t = 0, \quad a(t)|_{t=0} = a_0$$

and for $n \approx 1.0$ is represented by the function

$$a(t) = \sqrt{\frac{m_R}{k_R}(1 - e^{-2k_R t}) + a_0^2 e^{-2k_R t}},$$

tending asymptotically as $t \to \infty$ to

$$a_\infty \to \sqrt{\frac{m_R}{k_R}}. \tag{2a}$$

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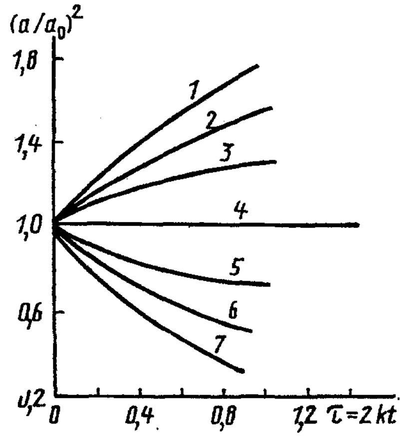

Fig. 1. Dependence of the mean droplet dimension on time for M = 3.0 (1), 2.0 (2), 1.5 (3), 0 (4), 0.5 (5), 0.2 (6), 0.05 (7).

Thus, if $a_0 > a_\infty = (m_R/k_R)^{1/2}$, then the droplet disintegration process predominates; otherwise (when $a_0 < a_\infty$) we observe their coagulation and growth in dimensions. As is seen from Fig. 1, the equilibrium between the coagulation and disintegration phenomena of droplets takes place at $M = m_R/(k_R a^2_0) = 1$. Consequently, if $M > 1$, we have the coagulation region; in this case when $M >> 1$ (to which large values of the turbulent diffusion coefficient and concentration of finely divided particles correspond) the growth of droplets dimensions occurs very intensely. When $M < 1$, the dimension of droplets decreases; naturally, in this region the disintegration phenomenon prevails.

Obviously, parameter k_R, characterizing the disintegration process of droplets in a turbulent flow, depends on the relation of the difference in dynamic heads (deforming a droplet) and the surface tension force, which is associated with the capillary pressure onset. Hence, if the capillary pressure exceeds the difference of the dynamic heads, then the droplet retains a stable shape. If we assume that intensive disintegration of droplets takes place, i.e., k_R is large ($M << 1$), then increase of droplets will lead to growth of the collision frequency and the rate of their coagulation.

We may describe the behavior of a great number of droplets in two ways: a) by applying the Fokker–Planck equations; b) by using critical equations for each type of droplet consisting of a definite number of molecules.

Therefore, it is quite possible to consider the droplet dimension as a continuously changing variable of Markov process and to use for its description the Fokker–Planck equation, which with account of (2) takes the form

$$\frac{\partial P}{\partial t} = -k_R \frac{\partial}{\partial a}\left[\left(\frac{m}{a} - a\right) P\right] + B\frac{\partial^2 P}{\partial a^2}, \quad m = m_R/k_R, \tag{3}$$

with the initial condition

$$P(a,\ t)|_{t=0} = P_0(a). \tag{3a}$$

The solution of (3) allows us to construct the evolution of the droplet dimension distribution in a turbulent flow, during which the normalization condition

$$\int_0^\infty P(a)\,da = 1.$$

is obeyed. To solve (3), we introduce the following expression:

$$P(a,\ t) = \varphi(t)\,\varphi(a), \tag{4}$$

and substituting it into (3), by means of separation of variables, we obtain two ordinary equations:

$$\frac{d\psi}{dt} = \mu\psi, \quad -k_R\frac{d}{da}\left[\left(\frac{m}{a} - a\right)\varphi(a)\right] + B\frac{d^2\varphi}{da^2} = \mu\varphi(a). \tag{5}$$

The solution of the first equation of (5) presents no difficulties:

$$\psi(t) = Ce^{\mu t}. \tag{6}$$

Introducing new variables $\theta = k_R m/B = m_R/B$, $y = k_R a^2/2B$, $\varphi = y^{\theta/2}e^{-y}\Phi(y)$, after simple but cumbersome transformations, we reduce the second equation of (5) to the form

$$y\frac{d^2\Phi}{dy^2} + \left(\frac{\theta+1}{2} - y\right)\frac{d\Phi}{dy} - \frac{\mu}{2k_R}\Phi = 0. \tag{7}$$

On the basis of Eqs. (6) and (7), it is possible to assume that (3) has the solution bounded at infinity, where $(-\mu/2k_R)$ is a whole number [5], i.e., $\mu = 2k_R n$ ($n = 0, 1, 2, \ldots$). Then the solution of (7) corresponding to eigenvalues μ is represented by Laguerre polynomials of n-th degree and $(\theta - 1)/2$ order, whereas the particular solution of (3) with account of (6) $\Phi(y) = \varphi y^{-\theta/2}e^{y}$ is represented as

$$P(a, t) = Ca^{\theta}\exp\left(-\frac{k_R a^2}{2B}\right)L_n^{(\alpha)}\left(\frac{k_R a^2}{2B}\right)e^{-2k_R nt},$$

where $\alpha = (\theta - 1)/2 = (m_R - B)/2B$, $L_n(a)$ is a Laguerre function. The general solution of (3) may be given in the form

$$P(a, t) = a^{\theta}\exp\left(-\frac{k_R a^2}{2B}\right)\sum_{n=0}^{\infty} C_n L_n^{(\alpha)}\left(\frac{k_R a^2}{2B}\right)e^{-2k_R nt}. \tag{8}$$

Using the initial condition (3a) and the orthogonality condition of the Laguerre function

$$\int_0^{\infty} e^{-a}a^{\alpha}L_m^{(\alpha)}(a)\,L_n^{(\alpha)}(a)\,da = \begin{cases}0, & m \neq n,\\ \Gamma(1+n+\alpha)\,n!, & m = n,\end{cases}$$

we determine the value for the constant coefficient C_n in (8) in the form

$$C_n = \frac{\theta^{\frac{\theta+1}{2}}\int_0^{\infty} P_0(a)\,L_n^{(\alpha)}(k_R a^2/2B)\,da}{m^{\theta+1}2^{\frac{\theta-1}{2}}\,\Gamma\left(n + \frac{\theta+1}{2}\right)n!}. \tag{9}$$

Thus, the solutions of Eqs. (8) and (9) make it possible to construct the dependence of the droplet dimension distribution function with account of their coagulation and disintegration on the duration of the action of the indicated physical phenomena, as well as on various parameters m_R, k_R and the initial distribution of droplets, characterizing the process. The asymptotic value of the distribution is obtained at $t \to \infty$ from (8), assuming that $L^{(\alpha)}_0(k_R a^2/2B) = 1$:

$$P_{\infty}(a, t) = C_0 a^{\theta}\exp\left(-\frac{k_R a^2}{2B}\right), \quad C_0 = 2\left(\frac{\theta}{2m}\right)^{\frac{\theta+1}{2}} = 2\left(\frac{k_R}{2B}\right)^{\frac{\theta+1}{2}}. \tag{10}$$

As follows from (10), for an infinite tube length a certain limiting distribution $P_{\infty}(a)$ is established, which is independent of the initial distribution. The mean limiting dimension of the droplets is written as

$$a_{\infty} = \int_0^{\infty} aP_{\infty}(a, t)\,da = C_0\int_0^{\infty} a^{\theta+1}\exp\left(-\frac{k_R}{2B}a^2\right)da = C_0\frac{\Gamma\left(\frac{\theta+2}{2}\right)}{2\left(\frac{k_R}{2B}\right)^{\frac{\theta+2}{2}}}.$$

Taking into consideration the value of C_0 from (10), we finally obtain

$$a_{\infty} = \sqrt{\frac{2B}{k_R}}\,\Gamma\left(1 + \frac{m_R}{2B}\right) = \sqrt{\frac{m_R^2}{2k_R B}}\,\Gamma\left(\frac{m_R}{2B}\right).$$

We introduce the relation

$$\frac{a_{\infty}}{a_0} = \sqrt{M\frac{m_R}{2B}}\,\Gamma\left(\frac{m_R}{2B}\right).$$

Hence, the relation $a_{\infty}/a_0 \sim M^{1/2}$ defines the predominance of coagulation or disintegration of droplets in the turbulent flow.

Proceeding from (10), we note that the limiting droplet distribution with respect to its form is close to the known empirical Rosen–Ramler relation, although depending on the parameter value, θ may be the Rayleigh distribution ($\theta = 1$), the Maxwell distribution ($\theta = 2$), etc.

We shall determine the coordinates for the maximal value of the limiting distribution from the condition

$$\frac{\partial P_\infty(a,\ t)}{\partial a} = C_0 \theta a^{\theta-1} \exp\left(-\frac{k_R a^2}{2B}\right) - C_0 a^\theta \frac{k_R a}{B} \exp\left(-\frac{k_R a^2}{2B}\right) = 0.$$

Reducing the parameters, being not equal to zero, we have the expression

$$a_\infty = (m_R/k_R)^{1/2},$$

which coincides with (2a).

Consequently, the maximal distribution value is shifted in the direction of smaller dimensions with decreasing m_R/k_R and in the direction of larger dimensions of droplets with increasing m_R/k_R. As follows from this expression, m_R and k_R are related through a_∞. It is possible to calculate the value of m_R by the formula $m_R = 4kD_T\varphi$, where the turbulent diffusion coefficient of particles D_T is determined from formulas given in [6]. Knowing the experimental value of a_∞, it is easy to obtain the estimate $k_R = m_g/a^2_\infty$.

NOTATION

a, droplet diameter; B, coefficient; D_T, turbulent diffusion coefficient; k, collision efficiency constant; l, length; $P(a, t)$, droplet distribution function; u_0, mean flow rate; μ_n, eigenvalues; φ, volumetric droplet fraction in a flow.

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